Integration using parameter

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It all begins when an oversea student sent a question to me by whatsapp:

> Show that $\int_1^e x \ln x \, dx = \frac{1}{4}(e^2+1)$.
>
> Hence, or otherwise, evaluate $\int_1^e x(\ln x)^2 dx$.

The solution is not too difficult. For the ***first part*** of the question

Method 1

Let $I = \int_1^e x \ln x \, dx$

Let $u = \ln x$, $e^u = x$, $e^u du = dx$

When $x = e, u = 1$. When $x = 1, u = 0$.

$\therefore I = \int_0^1 e^u (u) e^u du = \int_0^1 u e^{2u} du = \frac{1}{2}\int_0^1 u \, d(e^{2u}) = \frac{1}{2}\left[u e^{2u}|_0^1 - \int_0^1 e^{2u} du\right]$, integration by parts.

$= \frac{1}{2}\left[e^2 - \frac{e^{2u}}{2}\Big|_0^1\right] = \frac{1}{2}\left[e^2 - \left(\frac{e^2}{2} - \frac{1}{2}\right)\right] = \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{e^2} + \mathbf{1})$

Method 2

$\int_1^e x \ln x \, dx = \frac{1}{2}\int_1^e \ln x \, d(x^2) = \frac{1}{2}\left[x^2 \ln x|_1^e - \int_1^e x^2 d(\ln x)\right]$, integration by parts.

$= \frac{1}{2}\left[e^2 - \int_1^e x^2\left(\frac{dx}{x}\right)\right] = \frac{1}{2}\left[e^2 - \int_1^e x dx\right] = \frac{1}{2}\left[e^2 - \frac{x^2}{2}\Big|_1^e\right] = \frac{1}{2}\left[e^2 - \frac{e^2-1}{2}\right] = \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{e^2} + \mathbf{1})$

The ***second part*** of the question is also not too difficult.

Method 1

Let $I = \int_1^e x(\ln x)^2 dx$

Let $u = \ln x$, $e^u = x$, $e^u du = dx$

When $x = e, u = 1$. When $x = 1, u = 0$.

$\therefore I = \int_0^1 e^u (u^2) e^u du = \int_0^1 u^2 e^{2u} du = \frac{1}{2}\int_0^1 u^2 d(e^{2u}) = \frac{1}{2}\left[u^2 e^{2u}|_0^1 - 2\int_0^1 u e^{2u} du\right]$

$= \frac{1}{2}\left[e^2 - 2 \times \frac{1}{4}(e^2+1)\right]$, using the result of Method 1 in the first part.

$= \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{e^2} - \mathbf{1})$

Method 2

$\int_1^e x\,(\ln x)^2 dx = \frac{1}{2}\int_1^e (\ln x)^2\, d(x^2) = \frac{1}{2}\left[x^2(\ln x)^2|_1^e - \int_1^e x^2 d(\ln x)^2\right]$, integration by parts.

$= \frac{1}{2}\left[e^2 - \int_1^e x^2\left(2 \times \frac{1}{x}\ln x\, dx\right)\right] = \frac{1}{2}\left[e^2 - 2\int_1^e x\ln x\, dx\right]$

$= \frac{1}{2}\left[e^2 - 2 \times \frac{1}{4}(e^2+1)\right]$, using the result of Method 2 in the first part.

$= \mathbf{\frac{1}{4}(e^2 - 1)}$

The story begins here. Can we evaluate the integral by using higher power of ln x ?

We use, as illustration, $\int_1^e \mathbf{x\,(\ln x)^3 dx}$.

Let $u = \ln x$, $e^u = x$, $e^u du = dx$
When $x = e, u = 1$. When $x = 1, u = 0$.
We get

$$\boxed{\int_1^e \mathbf{x\,(\ln x)^3 dx} = \int_0^1 \mathbf{u^3 e^{2u} du}}$$

So we concentrate our discussion on $\int_0^1 \mathbf{u^3 e^{2u} du}$**.**

Let $f(t) = \int_0^1 e^{2tu} du = \left.\frac{e^{2tu}}{2t}\right|_0^1 = \frac{e^{2t}-1}{2t}$ ... (1)

$f'(t) = \frac{d}{dt}\int_0^1 e^{2tu} du = 2\int_0^1 u e^{2tu} du$ (note that we differentiate wrt to t, not u)

$f''(t) = \frac{d}{dt}\left[2\int_0^1 u e^{2tu} du\right] = 4\int_0^1 u^2 e^{2tu} du$

$f'''(t) = \frac{d}{dt}\left[4\int_0^1 u^2 e^{2tu} du\right] = 8\int_0^1 u^3 e^{2tu} du$, which is connected with our desired integral.

Now, we differentiate the RHS of (1),

$f'(t) = \frac{2te^{2t}-e^{2t}+1}{2t^2}$, $f''(t) = \frac{e^{2t}(2t^2-2t+1)-1}{t^3}$, $f'''(t) = \frac{e^{2t}(4t^3-6t^2+6t-3)+3}{t^4}$

Hence, $\int_0^1 e^{2tu} du = \frac{e^{2t}-1}{2t}$

$2\int_0^1 u e^{2tu} du = \frac{2te^{2t}-e^{2t}+1}{2t^2}$

$4\int_0^1 u^2 e^{2tu} du = \frac{e^{2t}(2t^2-2t+1)-1}{t^3}$

$8\int_0^1 u^3 e^{2tu} du = \frac{e^{2t}(4t^3-6t^2+6t-3)+3}{t^4}$

Put $t = 1$, we have all four integrals at the same time,

$$\int_0^1 e^{2u} du = \int_1^e \mathbf{x\,dx} = \frac{1}{2}(e^2 - 1)$$

$$\int_0^1 u e^{2u} du = \int_1^e \mathbf{x\,(\ln x)dx} = \frac{1}{4}(e^2 + 1)$$

$$\int_0^1 u^2 e^{2u} du = \int_1^e \mathbf{x\,(\ln x)^2 dx} = \frac{1}{4}(e^2 - 1)$$

$$\int_0^1 u^3 e^{2u} du = \int_1^e \mathbf{x\,(\ln x)^3 dx} = \frac{1}{8}(e^2 + 3)$$

Exercise

Show that (a) $\int_0^\infty x^5 e^{-2x}\, dx = \frac{15}{8}$

(b) $\int_0^1 \frac{x^{2017} - 1}{\ln x}\, dx = \ln(2018)$ by putting $f(t) = \int_0^1 \frac{x^t - 1}{\ln x}\, dx$.